

THE UNITED STATES OF AMERICA

TO ALL TO WHOM THESE PRESENTS SHALL COME:

**UNITED STATES DEPARTMENT OF COMMERCE
United States Patent and Trademark Office**

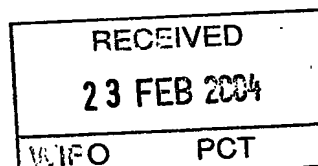
February 18, 2004

**THIS IS TO CERTIFY THAT ANNEXED HERETO IS A TRUE COPY FROM
THE RECORDS OF THE UNITED STATES PATENT AND TRADEMARK
OFFICE OF THOSE PAPERS OF THE BELOW IDENTIFIED PATENT
APPLICATION THAT MET THE REQUIREMENTS TO BE GRANTED A
FILING DATE.**


APPLICATION NUMBER: 60/433,407

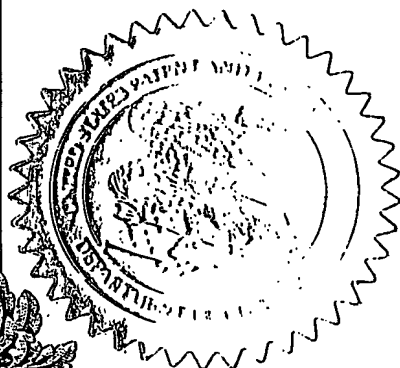
FILING DATE: *December 13, 2002*

RELATED PCT APPLICATION NUMBER: *PCT/US03/39714*



**By Authority of the
COMMISSIONER OF PATENTS AND TRADEMARKS**


**N. WILLIAMS
Certifying Officer**



**PRIORITY DOCUMENT
SUBMITTED OR TRANSMITTED IN
COMPLIANCE WITH
RULE 17.1(a) OR (b)**

BEST AVAILABLE COPY

12-16-02 0433407

A7 PROV

PROVISIONAL APPLICATION COVER SHEET

a request for filing a PROVISIONAL APPLICATION under 37 CFR § 1.53(c).

JC872 U.S. PRO
60/433407
12/13/02

EXPRESS MAIL LABEL NO:

EL884781637US

DATE OF DEPOSIT: Dec. 13, 2002

EL884781637US

Docket Number: UPN-4211

Type a plus
sign (+) inside
this box→JC872 U.S. PRO
60/433407
12/13/02

INVENTOR(S)/APPLICANT(S)

GIVEN NAME

(FIRST AND MIDDLE [IF ANY])

Charles L.

FAMILY NAME OR

SURNAME

Epstein

RESIDENCE (CITY AND EITHER STATE
OR FOREIGN COUNTRY)

Philadelphia, Pennsylvania

☐ Additional inventors are being named on the separately numbered sheets attached hereto

TITLE OF THE INVENTION (280 characters max)

MINIMUM ENERGY PULSE SYNTHESIS VIA THE INVERSE SCATTERING TRANSFORM

CORRESPONDENCE ADDRESS

Attorney Name: Michael P. Dunnam

WOODCOCK WASHBURN LLP

One Liberty Place - 46th Floor

Philadelphia PA 19103

Telephone (215) 568-3100

Facsimile (215) 568-3439

ENCLOSED APPLICATION PARTS (check all that apply)

☒ Specification Number of Pages: 38☐ Drawing(s) Number of Sheets:☒ Claims (optional)☐ Other (specify)

METHOD OF PAYMENT (check one)

☒ A check or money order is enclosed to cover the Provisional filing fee:☒ \$80.00 Small Entity☐ \$160.00 Large Entity☐ The Commissioner is hereby authorized to charge filing fee and credit Deposit Account Number: 23-3050☒ The Commissioner is hereby authorized to charge Deposit Account 23-3050 any fee deficiency or credit account for any overpayment.

The invention was made by an agency of the United States Government or under a contract with an agency of the United States Government.

☐ No.☒ Yes, the name of the U.S. Government agency and the Government contract numbers are NSF Grant Nos. DMS99-70487 and DMS02-07123.

Respectfully submitted,

SIGNATURE

TYPED or PRINTED NAME Michael P. Dunnam

Date: December 13, 2002
REGISTRATION NO. 32,611
(if appropriate)

PROVISIONAL APPLICATION FILING ONLY

Minimum Energy Pulse Synthesis via the Inverse Scattering Transform

Charles L. Epstein*
LSNI, Department of Radiology, HUP
and
Department of Mathematics, University of Pennsylvania

Abstract

This paper considers a variety of problems in the design of selective RF-pulses. We apply a formula of Faddeev and Zakharov to directly relate the energy of an RF-envelope to the magnetization profile and certain auxiliary parameters used in the inverse scattering transform approach to RF-pulse synthesis. This allows a determination of the minimum possible energy for a given magnetization profile. We give an algorithm to construct the minimum energy RF-envelope which includes an algorithm for solving the Gel'fand-Levitan-Marchenko equations.

Contents

1 The Bloch equation	3
2 Scattering theory for the ZS-system	6
3 Inverse scattering for the ZS-equation	10
4 The energy of the RF-envelope	13
5 Solving the Marchenko equation I	14
6 What is time?	19

*Research partially supported by NSF grants DMS99-70487 and DMS02-07123. Address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA. E-mail: cle@math.upenn.edu
Keywords: nuclear magnetic resonance, imaging, RF pulse synthesis, selective excitation, inverse scattering, Marchenko equation, minimum energy.

Underlying the derivation of the SLR algorithm is the so called *hard pulse approximation*. One designs a pulse envelope that is a finite sum of δ -pulses which produces an approximation to the desired magnetization profile. As δ -pulses are non-physical, requiring infinite energy, an SLR pulse envelope is implemented by replacing the train of δ -pulses by a train of boxcar pulses. The IST algorithm instead constructs a smooth pulse envelope which again produces an approximation to the desired magnetization profile. While such pulses are again implemented as a finite train of boxcar pulses, the approximations used in the IST are far less singular than those used in the SLR. Indeed the IST provides more direct and greater control over the final pulse envelope than

the SLR. When I undertook this work the central mystery for me was why Shinnar-Le Roux has remained the industry standard and the inverse scattering transform is largely ignored. In addition to presenting the results, which seem interesting in and of themselves, my second purpose in writing this paper is to present the IST approach in a more explicit and *palatable* form. This accounts, in part, for the somewhat expository nature of our presentation.

In the first and second sections we recall the connection between the Bloch equation without relaxation and their spin domain formulation. In inverse scattering the spin domain Bloch equation is called the Zakharov-Shabat 2×2 -system or ZS-system. We next review the scattering theory for the ZS-system and relate the scattering data to the magnetization profile. These sections closely follow [11], [10] and [1]. In the third section we outline the inverse scattering transform via the Gel'fand-Levitan-Marchenko equation, or more briefly the Marchenko equation, following the treatment in [1]. In the fourth section we state the formula relating the energy in the RF-envelope to the magnetization profile and the location of the bound states. In the fifth section we discuss the problem of implementing the IST. Under simplifying assumptions, we give a practical algorithm for solving the Marchenko equation. These assumptions are removed in Appendix A, where a modified algorithm is given which applies in all cases. In the sixth section we address the problem of rephasing and the physical significance of the time parameter. The seventh section contains several examples. In the eighth section we analyze the approximations used in both the SLR and IST approaches. Proofs of several mathematical results are contained in Appendix B.

1 The spin domain Bloch equation and the problem of RF-pulse synthesis

The Bloch equation without relaxation is usually written in the form

$$\frac{dM}{dt} = \gamma M \times B. \quad (1)$$

Here M is the magnetization, B is the applied magnetic field, t is time and γ is the gyromagnetic ratio. Since a vector evolves with constant length under this equation, the fundamental solution $U_0(t)$ can be regarded as $SO(3)$ -valued function of time. Here we normalize so that $U_0(0)$ is Id_3 , the 3×3 identity matrix. If $M(0) = M_0$ then $M(t) = U_0(t)M_0$. The solution to the Bloch equation is linear in the initial data. Throughout this paper we assume that solutions of the Bloch equation are normalized to have length equal to one.

1 THE BLOCH EQUATION

4

The Bloch equation, is usually analyzed in a "rotating reference" frame. Ordinarily the rotating reference frame is related to the "laboratory frame" by a time dependent orthogonal transformation of the form

$$F(t) = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) & 0 \\ \sin \theta(t) & \cos \theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$M(t) = F(t)m(t). \quad (2)$$

We use m to denote the magnetization in the rotating reference frame. Larmor's theorem implies that if M satisfies (1) then m satisfies

$$\frac{dm}{dt} = \gamma m \times B_{\text{eff}} \quad (3)$$

where

$$B_{\text{eff}}(t) = F^{-1}(t)B(t) + \frac{1}{\gamma}\Omega(t) \text{ with } \Omega(t) = [0, 0, \theta'(t)]^{\dagger}. \quad (4)$$

Notational remark: Most of the vectors used in this paper are to be thought of a column vectors. The notation $[a, b, c]^{\dagger}$ refers to the transpose of the row vector which is therefore a column vector, i.e.

$$[a, b, c]^{\dagger} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

In most applications of this method the function $\theta(t)$ is selected to render the z -component of B_{eff} independent of time,

$$B_{\text{eff}}(f; t) = (\omega_1(t), \omega_2(t), \gamma^{-1}f). \quad (5)$$

The constant value f is called the *offset frequency* or *resonance offset*. If we set $\tilde{\omega}(t) = \omega_1(t) + i\omega_2(t)$ then, in the laboratory frame, the RF-envelope takes the form:

$$p(t) = [\text{Re } \tilde{\omega}(t)e^{i\theta(t)}, \text{Im } \tilde{\omega}(t)e^{i\theta(t)}, 0]^{\dagger}.$$

The energy in the RF-envelope is given by

$$W_p = \int_{-\infty}^{\infty} |p(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{\omega}(t)|^2 dt. \quad (6)$$

The magnetization profile is a unit vector valued function defined for $f \in \mathbb{R}$,

$$m^{\infty}(f) = \begin{bmatrix} m_1^{\infty}(f) \\ m_2^{\infty}(f) \\ m_3^{\infty}(f) \end{bmatrix}.$$

1 THE BLOCH EQUATION

5

In essentially all MR applications $m^\infty(f) = [0, 0, 1]^\dagger$ for f outside of a bounded interval. The problem of RF-pulse synthesis is to find a time dependent complex pulse envelope $\tilde{\omega}(t)$ so that, if $B_{\text{eff}}(f)$ is given by (5), then the solution of

$$\frac{dm}{dt}(f; t) = \gamma m(f; t) \times B_{\text{eff}}(f; t) \quad (7)$$

with

$$\lim_{t \rightarrow -\infty} m(f; t) = [0, 0, 1]^\dagger \quad (8)$$

satisfies

$$\lim_{t \rightarrow \infty} [e^{-i\int_t} (m_1 + im_2)(f; t), m_3(f; t)] = [(m_1^\infty + im_2^\infty)(f), m_3^\infty(f)]. \quad (9)$$

We have used the standard complex notation for the transverse components of the magnetization. If $\tilde{\omega}$ is supported in the interval $[t_0, t_1]$ then these asymptotic conditions are replaced by

$$\begin{aligned} m(f; t_0) &= [0, 0, 1]^\dagger, \\ [e^{-i\int_{t_1}} (m_1 + im_2)(f; t_1), m_3(f; t_1)] &= [(m_1^\infty + im_2^\infty)(f), m_3^\infty(f)]. \end{aligned} \quad (10)$$

The mapping from ω to m^∞ is highly non-linear.

To solve the problem of RF-pulse synthesis it is convenient to introduce the spin domain formulation of the Bloch equation. Instead of a unit vector m in \mathbb{R}^3 we solve for a unit vector ψ in \mathbb{C}^2 . This vector satisfies the 2×2 matrix equation:

$$\frac{d\psi}{dt} = -\frac{i}{2} Q \cdot \sigma \psi. \quad (11)$$

Here $Q = -[\gamma\omega_1(t), \gamma\omega_2(t), f]$ and σ are the Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Assembling the pieces we see that ψ satisfies¹

$$\frac{d\psi}{dt}(\xi; t) = \begin{bmatrix} -i\xi & q(t) \\ -q^*(t) & i\xi \end{bmatrix} \psi(\xi; t), \quad (12)$$

with

$$\xi = \frac{f}{2}, \quad q = \frac{-i\gamma}{2}(\omega_1(t) - i\omega_2(t)).$$

A simple recipe takes a solution of (12) and produces a solution of (7). If $\psi(\xi; t) = [\psi_1(\xi; t), \psi_2(\xi; t)]^\dagger$ satisfies (12) then the 3-vector

$$m(f; t) = [2\text{Re}(\psi_1^* \psi_2), 2\text{Im}(\psi_1^* \psi_2), |\psi_1|^2 - |\psi_2|^2]^\dagger \left(\frac{f}{2}; t\right) \quad (13)$$

¹We follow the standard practice in the MR literature of using z^* to denote the complex conjugate of the complex number z .

2 SCATTERING THEORY FOR THE ZS-SYSTEM

6

satisfies (7). If in addition

$$\lim_{t \rightarrow -\infty} \psi(\xi; t) = \begin{bmatrix} e^{-i\xi t} \\ 0 \end{bmatrix}$$

then m satisfies (8). Thus the RF-pulse synthesis problem is easily translated into a inverse scattering problem for equation (12). This is described in the next section. Following the standard practice in inverse scattering we refer to equation (12) as the *ZS-system* and q as the *potential*.

2 Scattering theory for the ZS-system

Scattering theory for an equation like (12) relates the behavior of $\lim_{t \rightarrow -\infty} \psi(\xi; t)$ to that of $\lim_{t \rightarrow \infty} \psi(\xi; t)$. If q has bounded support then the functions

$$\begin{bmatrix} e^{-i\xi t} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^{i\xi t} \end{bmatrix}$$

are a basis of solutions for (12) outside the support of q . If the L^1 -norm of q is finite then it is shown in [1] that (12) has solutions which are asymptotic to these solutions as $t \rightarrow \pm\infty$.

Theorem 1. *If $\|q\|_{L^1}$ is finite then for every real ξ there are unique solutions*

$$\psi_{1+}(\xi), \psi_{2+}(\xi) \text{ and } \psi_{1-}(\xi), \psi_{2-}(\xi)$$

to equation (12) which satisfy

$$\lim_{t \rightarrow -\infty} e^{i\xi t} \psi_{1-}(\xi; t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lim_{t \rightarrow -\infty} e^{-i\xi t} \psi_{2-}(\xi; t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (14)$$

$$\lim_{t \rightarrow \infty} e^{i\xi t} \psi_{1+}(\xi; t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lim_{t \rightarrow \infty} e^{-i\xi t} \psi_{2+}(\xi; t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (15)$$

The solutions $\psi_{1-}(\xi), \psi_{2+}(\xi)$ extend as analytic functions of ξ to the upper half plane, $\text{Im } \xi > 0$ and $\psi_{2-}(\xi), \psi_{1+}(\xi)$ extend as analytic functions of ξ to the lower half plane, $\text{Im } \xi < 0$.

The proof of this theorem can be found in [1]. We briefly outline the construction of these solutions as we will need these formulae later on.

Outline of proof. The idea is to re-express the boundary value problems for the differential equation as Volterra integral equations. Let

$$\psi_{1-} = \begin{bmatrix} \psi_{11-} \\ \psi_{21-} \end{bmatrix}, \quad \psi_{2-} = \begin{bmatrix} \psi_{12-} \\ \psi_{22-} \end{bmatrix}.$$

These solutions satisfy:

$$\begin{aligned}
 \psi_{11-}(\xi; t)e^{i\xi t} &= 1 + \int_{-\infty}^t M_1(\xi; t, s)\psi_{11-}(\xi; s)e^{i\xi s} ds, \\
 \psi_{21-}(\xi; t)e^{i\xi t} &= - \int_{-\infty}^t e^{2i\xi(t-s)} q^*(s)\psi_{11-}(\xi; s)e^{i\xi s} ds, \\
 \psi_{22-}(\xi; t)e^{-i\xi t} &= -1 + \int_{-\infty}^t M_2(\xi; t, s)\psi_{22-}(\xi; s)e^{-i\xi s} ds, \\
 \psi_{12-}(\xi; t)e^{-i\xi t} &= - \int_{-\infty}^t e^{-2i\xi(t-s)} q(s)\psi_{22-}(\xi; s)e^{-i\xi s} ds,
 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
 M_1(\xi; t, s) &= -q^*(s) \int_s^t e^{2i\xi(x-s)} q(x) dx, \\
 M_2(\xi; t, s) &= -q(s) \int_s^t e^{2i\xi(s-x)} q^*(x) dx.
 \end{aligned} \tag{17}$$

The solutions normalized at $+\infty$ satisfy similar equations. If we set

$$Q_1(t) = \int_{-\infty}^t |q(s)| ds$$

then it is shown in [1] that these equations can be solved for ξ in the appropriate half plane by iteration and

$$|\psi_{11-}(\xi; t)e^{i\xi t}| \leq I_0(2Q_1(t)), \quad |\psi_{22-}(\xi; t)e^{-i\xi t}| \leq I_0(2Q_1(t)). \tag{18}$$

Here I_0 is the classical I_0 -Bessel function. The same argument applies *mutatis mutandis* to obtain ψ_{1+} and ψ_{2+} . \square

For real values of ξ the solutions normalized at $-\infty$ can be expressed in terms of the solutions normalized at $+\infty$ by the linear relations

$$\begin{aligned}
 \psi_{1-}(\xi; t) &= a(\xi)\psi_{1+}(\xi; t) + b(\xi)\psi_{2+}(\xi; t), \\
 \psi_{2-}(\xi; t) &= b^*(\xi)\psi_{1+}(\xi; t) - a^*(\xi)\psi_{2+}(\xi; t).
 \end{aligned} \tag{19}$$

The functions a, b are called the *scattering coefficients* for the potential q . The 2×2 -matrices $[\psi_{1-} \psi_{2-}]$, $[\psi_{1+} \psi_{2+}]$ satisfy

$$[\psi_{1-} \psi_{2-}] = [\psi_{1+} \psi_{2+}] \begin{bmatrix} a(\xi) & b^*(\xi) \\ b(\xi) & -a^*(\xi) \end{bmatrix}. \tag{20}$$

2 SCATTERING THEORY FOR THE ZS-SYSTEM

8

The scattering matrix for the potential q is defined to be

$$s(\xi) = \begin{bmatrix} a(\xi) & b^*(\xi) \\ b(\xi) & -a^*(\xi) \end{bmatrix}. \quad (21)$$

Recall that the *Wronskian* between two \mathbb{C}^2 -valued functions is defined by

$$W(u, v)(t) \stackrel{d}{=} u_1(t)v_2(t) - u_2(t)v_1(t).$$

If $u(t)$ and $v(t)$ are solutions of (12), for the same value of ξ , then $W(u(t), v(t))$ is independent of t . It is not difficult to show that

$$\begin{aligned} a(\xi) &= [\psi_{1-}(\xi; t)\psi_{22+}(\xi; t) - \psi_{21-}(\xi; t)\psi_{12+}(\xi; t)] = W(\psi_{1-}, \psi_{2+})(t), \\ b(\xi) &= -[\psi_{11-}(\xi; t)\psi_{21+}(\xi; t) - \psi_{21-}(\xi; t)\psi_{11+}(\xi; t)] = -W(\psi_{1-}, \psi_{1+})(t). \end{aligned} \quad (22)$$

It follows from (22) that a extends to the upper half plane as an analytic function. If q has an integrable derivative then, using (16) and (22) it can be shown that

$$a(\xi) = 1 + \frac{1}{2i\xi} \int_{-\infty}^{\infty} |q(s)|^2 ds + O\left(\frac{1}{\xi^2}\right) \quad (23)$$

as $|\xi|$ tends to infinity in $\text{Im } \xi \geq 0$. As $W(\psi_{1-}, \psi_{2-}) = -1$ it follows that

$$|a(\xi)|^2 + |b(\xi)|^2 = 1, \quad (24)$$

and therefore (23) implies that

$$|b(\xi)| = O\left(\frac{1}{|\xi|}\right) \text{ as } \xi \rightarrow \pm\infty. \quad (25)$$

These results are proved in [1].

Assuming that $t^j q(t)$ is integrable for all j it is shown in [1] that a has finitely many zeros in $\text{Im } \xi \geq 0$. Let (ξ_1, \dots, ξ_N) be a list of the zeros of a . For each j it follows from (22) that there is a non-zero complex number C_j so that

$$\psi_{1-}(\xi_j) = C_j \psi_{2+}(\xi_j), \quad j = 1, \dots, N. \quad (26)$$

The constants $\{C_1, \dots, C_N\}$ appearing in (26) are called *norming constants*. Equation (12) can be rewritten in the form

$$\begin{bmatrix} i\partial_t & -iq \\ -iq^* & -i\partial_t \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \xi \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \quad (27)$$

From this formulation it is clear that ξ should be regarded as a spectral parameter. If ξ has positive imaginary part then $\psi_{1-}(\xi; t)$ decays exponentially as t tends to $-\infty$ and $\psi_{2+}(\xi; t)$ decays exponentially as t tends to $+\infty$. If $a(\xi_j) = 0$ then (26) implies that $\psi_{1-}(\xi_j; t)$ decays exponentially at both $\pm\infty$ and therefore the function $\psi_{1-}(\xi_j; t)$

belongs to $L^2(\mathbb{R}; \mathbb{C}^2)$. Thus the operator on the left hand side of (27) has *bound states* for these values of the offset frequency.

We generally assume that the zeros of a are simple and that their imaginary parts are positive. This is mostly to simplify the exposition, there is no difficulty, in principle, if a has real zeros or higher order zeros. Using the argument principle and (23) it follows that

$$N = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{a'(\xi) d\xi}{a(\xi)}. \quad (28)$$

Definition 1. The pair of functions $(a(\xi), b(\xi))$, for $\xi \in \mathbb{R}$ and the collection of pairs $((\xi_j, C_j) : j = 1, \dots, N)$ define the *scattering data* for equation (12).

The scattering data are not quite independent. Since a is analytic in the upper half plane it follows from (24) and (23) that, for $\text{Im } \xi > 0$, we have:

$$a(\xi) = \prod_{j=1}^N \left(\frac{\xi - \xi_j}{\xi - \xi_j^*} \right) \exp \left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |b(\zeta)|^2) d\zeta}{\zeta - \xi} \right]. \quad (29)$$

see [7]. The *reflection coefficient* is defined by:

$$r(\xi) = \frac{b(\xi)}{a(\xi)}; \quad (30)$$

A priori the reflection coefficient is only defined on the real axis. We rewrite (29) in terms of r :

$$a(\xi) = \prod_{j=1}^n \left(\frac{\xi - \xi_j}{\xi - \xi_j^*} \right) \exp \left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\log(|r(\zeta)|^2) d\zeta}{\zeta - \xi} \right]. \quad (31)$$

Both (29) and (31) have well defined limits as ξ approaches the real axis.

Definition 2. The function $r(\xi)$, for $\xi \in \mathbb{R}$ and the collection of pairs $((\xi_j, C_j) : j = 1, \dots, N)$ define the *reduced scattering data* for equation (12).

Implicitly the reduced scattering data is a function of the potential q . In inverse scattering theory the data $\{r; (\xi_1, C_1), \dots, (\xi_N, C_N)\}$ are specified and we seek a potential q which has this reduced scattering data. The map from this data to q is often called the *Inverse Scattering Transform* or *IST*.

We close this section by rephrasing the RF-pulse synthesis problem as an inverse scattering problem. Recall that the data for the pulse synthesis problem is the magnetization profile m^{po} which we now think of as a function of $\xi = f/2$. Using (13), the solution ψ_{1-} to the ZS-system defines a solution m_+ to (7), satisfying (8). It follows from (15) and (19) that

$$\psi_{1-}(\xi; t) \sim \begin{bmatrix} a(\xi) e^{-i\xi t} \\ b(\xi) e^{i\xi t} \end{bmatrix} \text{ as } t \rightarrow +\infty. \quad (32)$$

3 INVERSE SCATTERING FOR THE ZS-EQUATION

10

Therefore

$$m_{1-}(\xi; t) \sim \left[\frac{2b(\xi)a^*(\xi)e^{2i\xi t}}{|a(\xi)|^2 - |b(\xi)|^2} \right], \quad (33)$$

where, as before, we use the complex notation for the transverse components of m_{1-} . If m_{1-} also satisfies (9) then it follows from (33) and (24) that

$$\begin{aligned} \frac{b(\xi)}{a(\xi)} &= \lim_{t \rightarrow \infty} \frac{(m_{11-} + im_{21-})(\xi; t)e^{-2i\xi t}}{1 + m_{31-}(\xi; t)} \\ &= \frac{(m_1^\infty + im_2^\infty)(\xi)}{1 + m_3^\infty(\xi)}. \end{aligned} \quad (34)$$

In light of this formula we sometimes refer to r as the *stereographic magnetization profile*. If q has support in the ray $(-\infty, t_1]$ then

$$r(\xi) = \frac{(m_{11-} + im_{21-})(\xi; t)e^{-2i\xi t}}{1 + m_{31-}(\xi; t)} \quad (35)$$

is independent of t for $t > t_1$.

As m^∞ is a unit vector valued function we see that the reflection coefficient $r(\xi)$ uniquely determines $m^\infty(\xi)$ and vice-versa. Thus the RF-pulse synthesis problem can be rephrased as the following inverse scattering problem:

Find a potential q for the ZS-system so that the reflection coefficient r satisfies (34) for all real ξ .

Note that the pulse synthesis problem makes no reference to the data connected to the bound states, i.e. $\{(\xi_j, C_j)\}$. Indeed these are *free* parameters in the pulse synthesis problem, making the problem highly underdetermined. Perhaps the main result of this paper is that, for a given magnetization profile, the RF-envelope requiring the minimum energy is the one for which the ZS-system has *no* bound states.

Remark 1. The material in this section is largely adapted from [1], [11] and [10].

3 Inverse scattering for the ZS-equation

There are several different approaches to solving the inverse scattering problem stated at the end of the previous section, see [1], [2], [4] or [7]. In this section we describe the solution of this problem via the Marchenko equations. We again follow the presentation in [1]. We generally suppose that the $\{\xi_j\}$ are distinct complex number with positive imaginary parts. To begin with we assume that the reflection coefficient is smooth and rapidly vanishing at $\pm\infty$. Using limiting arguments the latter restriction is easily relaxed.

The Marchenko equations are systems of integral equations of Fredholm type. Given the scattering data $\{r; (\xi_1, C_1), \dots, (\xi_N, C_N)\}$ define the function

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\xi) e^{i\xi t} d\xi - i \sum_{j=1}^N C_j e^{i\xi_j t}. \quad (36)$$

3 INVERSE SCATTERING FOR THE ZS-EQUATION

11

This is the inverse Fourier transform of r with a correction added to account for the bound states. Note that the finite sum is exponentially decreasing as t tends to $+\infty$. Define the 2×2 matrix valued function:

$$\mathcal{F}(t) = \begin{bmatrix} 0 & -f^*(t) \\ f(t) & 0 \end{bmatrix}.$$

For each number $t \in \mathbb{R}$ we seek a 2×2 -matrix valued function $\mathcal{H}_t(s)$, defined for $s \in [t, \infty)$, which solves the integral equation:

$$\mathcal{H}_t(s) + \int_t^\infty \mathcal{H}_t(\sigma) \mathcal{F}(s + \sigma) d\sigma = -\mathcal{F}(t + s) \text{ for } s \in [t, \infty). \quad (37)$$

Under our assumptions on r it is not difficult to show that, for every $t \in \mathbb{R}$, (37) is a Fredholm integral equation on $L^2([t, \infty); \mathfrak{M}_2)$. Since $\mathcal{F}(t + \cdot)$ belongs to $L^2([t, \infty); \mathfrak{M}_2)$, the solvability of (37) follows if it can be shown that any solution to the homogeneous adjoint equation,

$$\mathcal{L}(s) + \int_t^\infty \mathcal{L}(\sigma) \mathcal{F}^*(s + \sigma) d\sigma = 0$$

is identically zero. This proof of this statement can be found in [1]. The uniqueness of the solution to (37) follows from the fact that homogeneous equation corresponding to (37) also has only the trivial solution.

Remark 2. For the theory of Fredholm integral equations see [18]. For the moment we note that the known proofs for the existence of a solution to a Fredholm equation do not generally provide a practicable algorithm to find the solution.

The connection between the inverse scattering problem and the Marchenko equation is summarized in the following theorem:

Theorem 2. Given a smooth, rapidly decaying reflection coefficient r and a finite set of pairs $\{(\xi_j, C_j) : j = 1, \dots, N\}$, with the $\{\xi_j\}$ distinct, $\text{Im } \xi_j > 0$ and $C_j \neq 0$ for $j = 1, \dots, N$, the equation (37) has a unique solution for every $t \in \mathbb{R}$. If

$$q(t) = -2\mathcal{H}_{12t}(t) \quad (38)$$

then the ZS-system with this potential has reflection coefficient r . It has exactly N bound states for frequencies $\{\xi_1, \dots, \xi_N\}$ and the relations (26) hold at these points.

This theorem is proved in [7] and, in a slightly different formulation in [2].

Remark 3. As noted above, there is a similar result if a has higher order zeros or zeros on the real line. In this case the definition of f needs to be modified, we return to this in Appendix A.

We defer a discussion of the practicalities of solving (37) to section 5. For the moment suffice it to say that this equation can be solved by a simple iteration for values of t such that

$$\int_{2t}^\infty \|\mathcal{F}(\sigma)\| d\sigma < 1,$$

while a somewhat indirect method is required for smaller values of t . There is also "left" Marchenko equation, given in section 5, where the integrals are over half lines of the form $(-\infty, t]$. In this formulation the finite sum of exponentials is decreasing as t tends to $-\infty$.

The regularity and decay of q are related to the regularity and decay of r exactly as if they were a Fourier transform pair. The following result is a special case of a theorem proved in [2]:

Theorem 3 (Beals-Coifman). *Suppose that r is a reflection coefficient and q is the corresponding potential for the ZS-system. Let k be a positive integer.*

1. If

$$\int_{-\infty}^{\infty} [|\xi|^{2k} + 1] |r(\xi)|^2 d\xi < \infty$$

then the distributional derivatives $\partial_t^j q$ for $0 \leq j \leq k$ belong to $L^2(\mathbb{R}) + L^\infty(\mathbb{R})$.

2. If the distribution derivatives $\partial_\xi^j r \in L^2(\mathbb{R})$ for $0 < j \leq k+1$ then $t^{k+1} q \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$.

Remark 4. There are no smallness hypotheses in this theorem. Note also that this theorem does not mention the bound states. While the bound states exercise a strong effect on the detailed structure of q , their presence or absence has no effect on the gross smoothness and decay properties of the potential. These features of q are governed entirely by the reflection coefficient.

Remark 5. It has been noted many times in both the inverse scattering and MR literatures that, as q tends to zero, $r^*(\xi)$ tends to $\hat{q}(-2\xi)$. The results of Beals and Coifman shows that the connection between the inverse scattering transform and the Fourier transform is much deeper than that previously described. There is also an analogue of the Paley-Wiener theorem (see Appendix B): If the reflection coefficient has bounded support then the potential does not. This is easier to see in the logically equivalent contrapositive form: If the potential has bounded support then the reflection coefficient does not. When the potential has bounded support the scattering coefficients (a, b) extend, as analytic functions, to the entire complex plane. This means that the set where b and therefore r vanishes is discrete. Hence the reflection coefficient, restricted to the real axis, cannot have bounded support.

Remark 6. Beals and Coifman show that the IST can be extended to square integrable reflection coefficients such that $\partial_\xi r$ also belongs to $L^2(\mathbb{R})$. By using smooth, compactly support approximating sequences, the estimates in the theorem can be extended to data with finitely many derivatives and finite rates of decay.

The results of this section show that, at least in principle, the IST provides many solutions to the RF-pulse synthesis problem. A practical magnetization profile has bounded support and may be approximated by a smooth function. This leaves two essential problems:

4 THE ENERGY OF THE RF-ENVELOPE

13

1. Given m^∞ (and therefore r) how should the bound states be selected to give an "optimal" RF-pulse envelope which will produce the desired magnetization profile?
2. How is the Marchenko equation to be solved?

The notion of optimality depends on the intended application. It usually entails a balance among the energy, duration, oscillation and need for rephasing of the RF-envelope. In the remainder of the paper we focus on the construction of the minimum energy RF-envelope.

4 The energy of the RF-envelope

We now state a formula for the energy of the pulse envelope in terms of the scattering data. The underlying results are due to Zakharov, Faddeev and Manakov.

Theorem 4. Suppose that q is a sufficiently rapidly decaying potential for the ZS-system with reflection coefficient r and discrete data (ξ_j, C_j) , $j = 1, \dots, N$ then

$$\int_{-\infty}^{\infty} |q(t)|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log(1 + |r(\xi)|^2) d\xi + 4 \sum_{j=1}^N \operatorname{Im} \xi_j. \quad (39)$$

The proof of this result can be found in [1] or [7]. Note that the norming constants play no role in this formula. Formula (39) is just one from an infinite sequence of formulae relating functionals of the potential to functionals of the reduced scattering data.

Combining (6), (39) with (34) we obtain the following simple corollary.

Corollary 1. If m^∞ is a sufficiently smooth magnetization profile such that $m_1^\infty + im_2^\infty$ vanishes outside a finite interval then the total energy of any RF-envelope ω which produces this magnetization profile satisfies the estimate

$$\int_{-\infty}^{\infty} |\omega(t)|^2 dt \geq \frac{2}{\pi \gamma^2} \int \log \left(1 + \left| \frac{m_1^\infty(\xi) + im_2^\infty(\xi)}{1 + m_3^\infty(\xi)} \right|^2 \right) d\xi. \quad (40)$$

Equality holds in this estimate if and only if the ZS-system with the corresponding potential has no bound states.

From the corollary it is evident that the lowest energy RF-envelope is obtained by solving the Marchenko equation with

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\xi) e^{i\xi t} d\xi.$$

5 SOLVING THE MARCHENKO EQUATION I

14

Remark 7. There are several other immediate conclusions which can be drawn from this formula. Often times one seeks to specify a particular dependence for the phase of the transverse magnetization. Changing the phase of the transverse magnetization amounts to replacing $(m_1^\infty(\xi) + im_2^\infty(\xi))$ by $(m_1^\infty(\xi) + im_2^\infty(\xi))e^{i\phi(\xi)}$, where ϕ is a real valued function. From (40) it is apparent that the energy required for the minimum energy RF-envelope is independent of the phase of the transverse magnetization. Note however that the minimum energy pulse will in general require rephasing. This is addressed in section 6.

Remark 8. Very important types of pulses in MR imaging are inversion or refocussing pulses. Such a pulse carries $[0, 0, 1]$ to $[0, 0, -1]$. If this is the case at offset frequency ξ then $r(\xi) = \infty$. It would therefore require infinite energy to exactly invert $[0, 0, 1]$ for offset frequencies belonging to an interval of positive length. The energy required to carry $[0, 0, 1]$ to a vector of the form $[m_1, m_2, -1 + \epsilon]$ for $0 \leq \epsilon < \epsilon$ and offset frequencies belonging to a band of width B , is at least

$$\frac{2B}{\pi\gamma^2} \log \left(1 + \frac{4(2 - \epsilon)}{\epsilon} \right).$$

Ideally we would like to have a functional that estimates the support of q in terms of the scattering data. An explicit formula of this sort has proved elusive. On the other hand, as we shall see in section 5, the Marchenko equation implies that the *effective* support of q is largely determined by the effective support of the function f defined in (36).

In the absence of bound states it is not hard to show that if $r(\xi)$ is the reflection coefficient determined by $q(t)$ then $r(\lambda\xi)$ comes from the potential $\lambda^{-1}q(\lambda^{-1}t)$. This agrees with the well known heuristic principle that the effective support of the RF-envelope is inversely proportional to the bandwidth of excitation. It also shows that the maximum amplitude and energy of the RF-envelope are proportional to the bandwidth. The L^2 -oscillation of the RF-envelope, $\int |q_t|^2 dt$, is proportional to the square of the bandwidth.

5 Solving the Marchenko equation with no bound states

In this section we discuss an algorithmic approach for obtaining the minimum energy RF-envelope. It entails solving the Marchenko equation, (37) and a related equation, (42) given below. Our method involves solving the Marchenko equations directly via iteration. As noted above, this requires some care as large data may lead to a divergent series. In [11] Rourke and Morris introduced a method for solving the Marchenko equation by adapting a method of Moses and Proesser, see [9]. The analysis in [11] differs in one small particular from that in [9], leading to solutions of the Marchenko equation with non-trivial bound states and therefore sub-optimal energy requirement.

The method of Moses and Proesser is to approximate the reflection coefficient by a rational function with all of its poles in the lower half plane. This ensures that the resultant potential is supported in $(-\infty, 0]$ and has no bound states. In [11] this condition on the placement of the poles is not respected. Indeed in examples 2 and 3 of [11]

5 SOLVING THE MARCHENKO EQUATION I

15

they use a rational approximation to r with poles in both half planes; the solutions they found were supported in a proper subinterval of $(-\infty, 0)$. However the poles of r in the upper half plane force the resultant potential for the ZS-system to have non-trivial bound states. These increase the needed energy and oscillation without affecting the magnetization profile. Regrettably the method of Moses and Proesser is not generally applicable to find the minimum energy RF-envelope. This is due to the well known consequence of the maximum principle in the theory of functions of a complex variable: A function on the real line which tends to zero at infinity is only well approximated by rational functions with poles in the lower half plane if it is the boundary value of an analytic function defined in the upper half plane. In light of a classical theorem of Fatou, this excludes any function supported in a bounded interval, see [13].

We are left with little choice but to solve the Marchenko equation directly. We now present an algorithm for doing this. The algorithm is described in the realm of continuum mathematics but can easily be approximately implemented on a computer. We will return to the question of implementation in a later publication. After stating the "left" Marchenko equation we give an estimate which shows that, for a given reflection coefficient, the Marchenko equations can be solved for sufficiently large $|t|$. There are two ideas which underlie our algorithm:

1. Using the Marchenko equation we solve for the potential *one t at a time*.
2. If $q = q_1 + q_2$ where q_1 and q_2 have disjoint supports then the scattering data for q has a simple, explicit expression in terms of the scattering data for q_1 and q_2 .

Using these ideas leads to a method which allows us to work inward from $\pm\infty$ to determine the potential for all values of t .

Theorem 2 and its extension to less regular data imply that, given a sufficiently regular reflection coefficient r , there exists a potential q so that the ZS-system, with this potential, has reflection coefficient r and no bound states. In this and the following section the data r and the corresponding potential q are fixed. We assume that r decays rapidly enough and is sufficiently regular for q to be bounded and integrable with two continuous derivatives, see Theorem 3.

Equation (37) involves integration over positive half rays. Because of the divergences indicated above it is useful to work both ends against the middle. For that purpose we give the *left Marchenko equation* which involves integration over negative half rays. For later applications we give the left Marchenko equation with finitely many bound states. Recall that $r = b/a$; given r we can use (31) to determine a and therefore b . Define

$$\tilde{r}(\xi) = \frac{b^*(\xi)}{a(\xi)}$$

and let $\{\xi_1, \dots, \xi_N\}$ be points in the upper half plane. There are non-zero constants $\{\tilde{C}_j\}$ so that the kernel function for the left Marchenko equation is defined by

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{r}(\xi) e^{-i\xi t} d\xi - i \sum_{j=1}^N \tilde{C}_j e^{-i\xi_j t} \quad (41)$$

The exponential correction terms are exponentially decaying as t tends to $-\infty$. The kernel for the left Marchenko equation is

$$\mathcal{G}(t) = \begin{bmatrix} 0 & g(t) \\ -g^*(t) & 0 \end{bmatrix}.$$

For each fixed $t \in \mathbb{R}$ the equation is

$$\mathcal{L}_t(s) + \int_{-\infty}^t \mathcal{L}_t(x) \mathcal{G}(x+s) dx = -\mathcal{G}(t+s) \text{ for } s \in (-\infty, t]. \quad (42)$$

If $\mathcal{L}_t(s)$ solves this equation then the potential defined by

$$q(t) = 2\mathcal{L}_t(t)$$

has reflection coefficient r and bound states at $\{\xi_j\}$. In the remainder of our analysis we concentrate on the "right" Marchenko equation, (37), with the understanding that everything said applies *mutatis mutandis* to the left formulation as well.

Let $\|\cdot\|$ denote an operator norm on 2×2 -matrices; this means that $\|\cdot\|$ is a norm which satisfies:

$$\|AB\| \leq \|A\| \|B\|$$

For example we can take

$$\|A\| = \max_{v \neq 0} \frac{\|Av\|}{\|v\|}.$$

Define the functions

$$\begin{aligned} M_f(t) &= \sup_{s \geq t} \|\mathcal{F}(s)\|, \\ I_f(t) &= \int_t^\infty \|\mathcal{F}(s)\| ds. \end{aligned} \quad (43)$$

We say that the Marchenko equation can be *solved by iteration* on the interval $[t, \infty)$ if the sequence defined by

$$\begin{aligned} \mathcal{H}_t^0(s) &= -\mathcal{F}(s+t), \\ \mathcal{H}_t^j(s) &= -\mathcal{F}(s+t) + \int_t^\infty \mathcal{H}_t^{j-1}(s+x) \mathcal{F}(t+x) dx, \quad j = 1, 2, \dots \end{aligned} \quad (44)$$

converges uniformly to $\mathcal{H}_t(s)$ for $s \in [t, \infty)$.

Theorem 5. *If $I_f(2t) < 1$ then the Marchenko equation can be solved by iteration on the interval $[t, \infty)$. The solution satisfies the estimate*

$$\mathcal{H}_t(s) \leq \frac{M_f(t+s)}{1 - I_f(2t)}. \quad (45)$$

5 SOLVING THE MARCHENKO EQUATION I

17

The proof is given in Appendix B. Note that no assumption is made here about the presence or absence of bound states.

The estimate (45) implies that, so long as the hypotheses of the theorem hold, the potential q satisfies the estimate

$$|q(t)| \leq \frac{2M_f(2t)}{1 - I_f(2t)}. \quad (46)$$

This in turn explains our claim that the essential support of q is determined by the essential support of f . A similar result holds for the left Marchenko equation. Taken together they show that if r decays rapidly enough and is sufficiently smooth then the potential q can be determined, by iteration outside of a bounded interval $[t_-, t_+]$. If $I_f(-\infty) < 2$ then q can be determined on the whole real line by iteration.

If $I_f(-\infty) \geq 2$ then an additional trick may be needed. If we rewrite the potential q as a sum of potentials with disjoint support:

$$\begin{aligned} q &= \chi_{(-\infty, t_0]} q + \chi_{(t_0, \infty)} q \\ &= q_1 + q_2 \end{aligned} \quad (47)$$

then the scattering matrix of q is easily expressed in terms of the scattering matrices of q_1 and q_2 . Indeed we have two formulae.

Theorem 6. Suppose that $q = q_1 + q_2$ where the support of q_1 is to the left of the support of q_2 . If

$$s_i = \begin{bmatrix} a_i & b_i^* \\ b_i & -a_i^* \end{bmatrix}, \quad i = 1, 2,$$

are the scattering matrices defined by q_i , $i = 1, 2$, respectively then the scattering matrix defined by q is given by

$$\begin{aligned} \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} &= \begin{bmatrix} a_2 & -b_2^* \\ b_2 & a_2^* \end{bmatrix} \begin{bmatrix} a_1 & b_1^* \\ b_1 & -a_1^* \end{bmatrix} \\ &= \begin{bmatrix} a_2 & b_2^* \\ b_2 & -a_2^* \end{bmatrix} \begin{bmatrix} a_1 & b_1^* \\ -b_1 & a_1^* \end{bmatrix} \end{aligned} \quad (48)$$

These formulae are elementary consequences of the definition of the scattering matrix.

The abstract existence theory for the Marchenko equation assures that, given a sufficiently smooth, square integrable reflection coefficient r , there are solutions of the corresponding Marchenko equations for every $t \in \mathbb{R}$ and therefore a potential q which defines a ZS-system with no bound states and the given reflection coefficient. In light of Theorem 5 there are numbers $\{t_-^{(0)}, t_+^{(0)}\}$ so that we can solve the left and right Marchenko equations by iteration to find

$$\begin{aligned} q_-^{(0)} &= \chi_{(-\infty, t_-^{(0)})} q \text{ and} \\ q_+^{(0)} &= \chi_{(t_+^{(0)}, \infty)} q. \end{aligned} \quad (49)$$

If $t_-^{(0)} \geq t_+^{(0)}$ then we are done. Otherwise we need to solve the forward scattering problem for the potentials $q_-^{(0)}$ and $q_+^{(0)}$ to obtain the scattering coefficients $\{(a_{\pm}^{(0)}, b_{\pm}^{(0)})\}$. Using Theorem 6 twice we obtain the relation:

$$\begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} = \begin{bmatrix} a_+^{(0)*} & -b_+^{(0)*} \\ b_+^{(0)} & a_+^{(0)} \end{bmatrix} \begin{bmatrix} a_1 & b_1^* \\ b_1 & -a_1^* \end{bmatrix} \begin{bmatrix} a_-^{(0)} & b_-^{(0)*} \\ -b_-^{(0)} & a_-^{(0)*} \end{bmatrix} \quad (50)$$

Here (a_1, b_1) are the scattering coefficients for the (unknown part of the) potential

$$q_1 = \chi_{[t_-^{(0)}, t_+^{(0)}]} q. \quad (51)$$

Using (50) we solve for the scattering coefficients of q_1 . Since q_1 is a potential with bounded support the functions (a_1, b_1) extend as analytic functions to the entire complex plane. It is not clear whether or not the ZS-system with potential q_1 has any bound states. In fact our algorithm applies in either case. To simplify the exposition we assume, for now, that for all $\alpha < \beta$ the potentials $\chi_{[\alpha, \beta]} q$ have no bound states. In Appendix A we explain how to remove this hypothesis. The iterative step in the algorithm relies on the following Paley-Wiener type theorem:

Theorem 7. Suppose that q is a potential for the ZS-system with support in the halfline $(-\infty, t_+]$ ($[t_-, \infty)$). Then the kernel function f for the Marchenko equation, defined in (36), (g defined in (41)) is supported in the ray $(-\infty, 2t_+]$ ($[2t_-, \infty)$).

The proof of this result is given in Appendix A.

Assuming that q_1 has no bound states, the pair of functions (a_1, b_1) is the scattering data for a potential with support in the finite interval $[t_-^{(0)}, t_+^{(0)}]$. Theorem 7 therefore applies to show that

$$f_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r_1(\xi) e^{i\xi t} d\xi \quad (52)$$

is a function with support in the interval $(-\infty, 2t_+^{(0)}]$, with an analogous conclusion for g_1 .

From Theorem 5 it follows that there exists another pair $(t_-^{(1)}, t_+^{(1)})$ such that

$$t_-^{(0)} < t_-^{(1)}, \quad t_+^{(1)} < t_+^{(0)}$$

and the Marchenko equations, with the kernels defined by g_1, f_1 , can be solved by iteration on the intervals $(-\infty, t_-^{(1)}]$ and $[t_+^{(1)}, \infty)$ respectively. Note that the potential q is now determined on the complement of $[t_-^{(1)}, t_+^{(1)}]$. Next the scattering data for the potentials

$$\begin{aligned} q_-^{(1)} &= \chi_{[t_-^{(0)}, t_-^{(1)}]} q \text{ and} \\ q_+^{(0)} &= \chi_{[t_+^{(1)}, t_+^{(0)}]} q. \end{aligned} \quad (53)$$

are determined. Using Theorem 6 this data can be used to determine (a_2, b_2) , the scattering data for $q_2 = \chi_{[t_-^{(1)}, t_+^{(1)}]} q$.

6 WHAT IS TIME?

19

The method is applied iteratively to obtain sequences

$$t_-^{(0)} < \dots < t_-^{(j)} < t_-^{(j+1)} < \dots < t_+^{(j+1)} < t_+^{(j)} < \dots < t_+^{(0)}.$$

In the Appendix A we show that, given sufficiently smooth initial data r there is $\Delta > 0$ so that, for all $j \geq 0$,

$$|t_{\pm}^{(j)} - t_{\pm}^{(j+1)}| > \Delta.$$

Hence after finitely many steps the potential q is determined on the whole real line.

We close this section by stating our algorithm as a sequence of steps. In the next section we indicate the modifications needed to handle the possible occurrence of bound states for the "intermediate" potentials $\{q_j\}$.

STEP 0:

Solve the left and right Marchenko equations by iteration on "maximal" intervals $(-\infty, t_-^{(0)}]$, $[t_+^{(0)}, \infty)$. If $t_-^{(0)} \geq t_+^{(0)}$ we are done. Otherwise determine the scattering data for $q_-^{(0)}$ and $q_+^{(0)}$. Apply Theorem 6 to determine the scattering data for q_1 . Set $j = 1$ and proceed to step 1.

STEP 1:

Use the reflection coefficient for q_j in (52) to determine the kernel function $f_j(g_j)$ supported on $(-\infty, 2t_+^{(j-1)}]$ ($[2t_-^{(j-1)}, \infty)$).

STEP 2:

Solve the Marchenko equation on the interval $[t_+^{(j)}, t_+^{(j-1)}]$ ($[t_-^{(j-1)}, t_-^{(j)}]$). If $t_-^{(j)} \geq t_+^{(j)}$ we are done, otherwise go to step 3.

STEP 3:

Determine the scattering data, on the real line for the potentials

$$q_-^{(j)} = \chi_{[t_-^{(j-1)}, t_-^{(j)}]} q, \quad q_+^{(j)} = \chi_{[t_+^{(j)}, t_+^{(j-1)}]} q$$

and apply Theorem 6 to determine the scattering data, on the real line, for

$$q_{j+1} = \chi_{[t_-^{(j)}, t_+^{(j)}]} q.$$

STEP 4: Go to step 1 with j replaced by $j+1$.

6 What is time?

Before we turn to the computation of examples we need to consider the physical significance of the t -parameter which appears in (11). This is intimately connected to the problem of rephasing. The main clue comes from (35). Suppose that q is a potential supported in $[t_0, t_1]$ with reflection coefficient r . After time $= t$ a solution to (11) is freely precessing in the local (gradient offset) B_0 -field.

6 WHAT IS TIME?

20

If $t_1 = 0$ then, at the end of the RF-excitation, the magnetization satisfies

$$\frac{(m_{11-} + im_{21-})(\xi; 0)}{1 + m_{31-}(\xi; 0)} = r(\xi). \quad (54)$$

We have therefore achieved the desired magnetization profile without any need for rephasing. If on the other hand $t_1 < 0$ then, at the end of the excitation, the magnetization satisfies

$$\frac{(m_{11-} + im_{21-})(\xi; t_1)}{1 + m_{31-}(\xi; t_1)} = e^{2i\xi t_1} r(\xi). \quad (55)$$

In order to get the desired magnetization profile the spins must freely precess for $|t_1|$ -units of time. Finally if $t_1 > 0$ then the spins must precess for $-t_1$ -units of time. In other words either a 180° -refocussing pulse must be applied or the gradient needs to be reversed, followed by t_1 -units of free precession, in order to achieve the magnetization profile specified by r .

In RF-pulse synthesis the data is specified in the *frequency* domain. Given $r(\xi)$ and perhaps some bound states $\{(\xi_j, C_j)\}$ the IST produces a potential which is a function of t . The support of this potential is determined by the scattering data and thereby determines what sort of rephasing is needed to achieve the specified magnetization profile. If the *essential* support of the potential is contained in $[0, t_1]$ then, according to the sign of t_1 , the necessary rephasing is determined as in the previous paragraph. The main point to note is that the time dependence of the potential is an *output* of the IST. In particular, the numerical value of t_1 , in appropriate units, determines the rephasing which is needed to achieve the specified magnetization profile. When computing the total energy requirement for an RF-envelope with $t_1 > 0$ one must include the energy of the (hard) 180° -pulse or that of transients produced by reversing the gradient.

Two tools are available to manipulate the support of q . On the one hand we can use smooth approximations to r so that its Fourier transform is rapidly decreasing. The estimate (46) shows that this reduces the essential support of q . The other tool is the introduction of bound states, coupled with methods of meromorphic approximation. Rourke and Morris approximate r by rational functions which decay at infinity and obtain potentials with support in $(-\infty, 0]$. The rational approximations to r which they use have poles in the upper half plane and these in turn lead to an increase in the energy and the oscillation of the RF-envelope. This is the price for an envelope that does not require rephasing. For the standard pulses used in MR imaging the minimum energy pulse in general requires rephasing.

The problem of finding approximations that carefully balance the rephasing requirements with the energy of the pulse itself seems a very interesting field for further investigation. Through the Marchenko equations and the Paley-Wiener Theorem II it translates to the problem of approximating the function r , defined on real axis, by meromorphic functions ρ , defined in the upper half plane, which satisfy an estimate of the form

$$\int_{-\infty}^{\infty} e^{-4\eta T} |\rho(\xi + i\eta)|^2 d\xi \leq C \text{ for } \eta \geq 0.$$

Here C and T are positive constants. The resultant potential will be supported in $(-\infty, T]$. We will return to this variational problem in a subsequent publication.

7 Examples

8 Analysis of the hard pulse approximation

The Shinnar-Le Roux algorithm uses the *hard pulse approximation* to design an RF-envelope. Given a magnetization profile $r(\xi)$ the SLR-algorithm designs a potential of the form

$$q_0 = \sum_{j=1}^N \mu_j \delta(t - j\Delta),$$

to produce a good approximation to the desired profile over a sub-interval of

$$[-\Delta^{-1}\pi, \Delta^{-1}\pi].$$

The reflection coefficient, η defined by q_0 is a *periodic* function of period $\Delta^{-1}2\pi$. The SLR-algorithm provides an iterative method for determining the coefficients $\{\mu_j\}$.

Of course a sum of δ -pulses is non-physical, requiring infinite energy to realize. The RF-envelope which is actually used is a "softened" version of q_0 . For example one could replace each $\mu_j \delta(t - j\Delta)$ by a boxcar pulse of width Δ with the same area, leading to the softened pulse:

$$q_1 = \sum_{j=1}^N \frac{\mu_j}{\Delta} \chi_{[0, \Delta)}(t - j\Delta),$$

see [14]. There is no reasonable sense in which the difference $q_0 - q_1$ is "small." Nonetheless, under certain conditions, the difference $|r_0(\xi) - r_1(\xi)|$ can be made small for ξ in fixed interval. The reflection coefficient, r corresponding to q_1 tends to zero at infinity. The relationship between η and r_1 is studied below.

The SLR approach should be contrasted with the approach outlined above: Given r we approximate it by a smooth magnetization profile r_s . We then construct a smooth potential q_s with reflection coefficient r_s . Such a pulse is again implemented by replacing it with a sum like q_1 . So there should be no confusion, we denote a sequence of boxcar pulses approximating q_s by q_{s1} . Below we show that the difference in the magnetization profile produced by q_s and that produced by q_{s1} is controlled by $\|q_{s1} - q_s\|_{L^1}$.

For our analysis of the hard pulse approximation we consider the one parameter family of pulses

$$q_\epsilon = \sum_{j=1}^N \frac{\mu_j}{\epsilon\Delta} \chi_{[0, \epsilon\Delta)}(t - j\Delta), \quad \epsilon \in [0, 1], \quad (56)$$

with the understanding that q_0 is the sum of δ -pulses given above. It is a simple computation to show that

$$\int_{-\infty}^{\infty} |q_\epsilon(t)| dt$$

8 ANALYSIS OF THE HARD PULSE APPROXIMATION

22

is independent of ϵ and

$$\int_{-\infty}^{\infty} |q_{\epsilon}(t)|^2 dt = \frac{1}{\epsilon} \int_{-\infty}^{\infty} |q_1(t)|^2 dt.$$

This formula indicates the desirability of using q_1 to approximate q_0 rather than q_{ϵ} for an $\epsilon < 1$.

Let r_{ϵ} denote the reflection coefficient defined by q . The potential q_{ϵ} is supported in the interval $[0, N\Delta]$. In order to study r_{ϵ} we need to find a formula for $\psi_{\epsilon 1-}(\xi; N\Delta)$. As is well known, this can be conveniently expressed in terms of a product of 2×2 -matrices. The solution operator at time $t_0 + \Delta$ to the ZS-system with potential given by

$$\epsilon^{-1} \mu \chi_{[0, \epsilon \Delta]}(t - t_0)$$

is

$$U_{\mu, \epsilon, \Delta}(\xi) = \begin{bmatrix} e^{-i(1-\epsilon)\Delta\xi} \left(\cos(\alpha) - \frac{i\epsilon\Delta\xi \sin(\alpha)}{\alpha} \right) & e^{-i(1-\epsilon)\Delta\xi} \left(\frac{i\mu \sin(\alpha)}{\alpha} \right) \\ e^{i(1-\epsilon)\Delta\xi} \left(\frac{i\mu^* \sin(\alpha)}{\alpha} \right) & e^{i(1-\epsilon)\Delta\xi} \left(\cos(\alpha) + \frac{i\epsilon\Delta\xi \sin(\alpha)}{\alpha} \right) \end{bmatrix}, \quad (57)$$

where

$$\alpha = \sqrt{\xi^2 + |\mu|^2}. \quad (58)$$

The solution operator for the ZS-system, at time $N\Delta$ with the potential given by (56) is therefore the product of the 2×2 -matrices:

$$\begin{aligned} U_{\epsilon, \Delta}(\xi) &= U_{\mu_N, \epsilon, \Delta}(\xi) \cdots U_{\mu_1, \epsilon, \Delta}(\xi) \\ &= \begin{bmatrix} A_{\epsilon}(\xi) & -B_{\epsilon}^*(\xi) \\ B_{\epsilon}(\xi) & A_{\epsilon}^*(\xi) \end{bmatrix}. \end{aligned} \quad (59)$$

It follows easily from (34) that

$$r_{\epsilon}(\xi) = \frac{B_{\epsilon}(\xi) e^{-2iN\Delta\xi}}{A_{\epsilon}(\xi)}. \quad (60)$$

The entries of (57), (59) and (57) are continuous at $\epsilon = 0$. The functions $(A_{\epsilon}, B_{\epsilon})$ are the scattering coefficients for the ZS-system with potential q_{ϵ} . If $\epsilon \neq 0$ the asymptotics established in (65) are applicable. If $\epsilon = 0$ then A_0 and B_0 can be expressed as

$$A_0(\xi) = e^{iM\Delta\xi} P(e^{i\Delta\xi}), \quad B_0(\xi) = e^{iM'\Delta\xi} Q(e^{i\Delta\xi}),$$

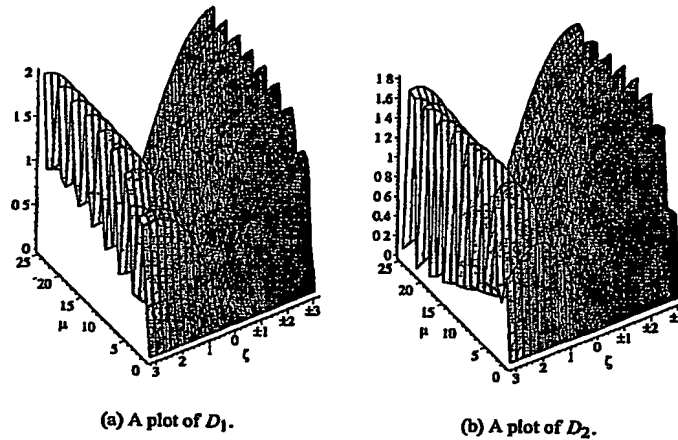
Where P and Q are polynomials of degree $N-1$. As expected A_0 and B_0 are $\Delta^{-1}2\pi$ -periodic functions.

The SLR design process begins with $r_0(\xi)$, a rational function of an appropriate degree in $e^{i\Delta\xi}$, approximating the desired magnetization profile $r(\xi)$ over the interval $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$. The algorithm then determines coefficients $\{\mu_j\}$ so that

$$r_0(\xi) \approx \frac{B_0(\xi) e^{-2iN\Delta\xi}}{A_0(\xi)}.$$

8 ANALYSIS OF THE HARD PULSE APPROXIMATION

23

Figure 1: Plots of the error functions D_1 and D_2 for $\mu \in [0, 25]$.

As with the IST approach this problem does not have a unique solution. The different solutions are labeled, in essence, by the locations of the zeros of P outside the unit disk. These correspond to zeros of A_0 in the upper half plane. The questions of principal interest are:

1. How well does $r_1(\xi)$ approximate $r_0(\xi)$ over the interval $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$?
2. How rapidly does r_1 decay outside of $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$?
3. How are the zeros of A_1 in the upper half plane related to those of A_0 ?

The first question admits of a fairly simple analysis. We first consider a single term in the product expansion (59), examining the dependence of the difference $\|U_{\mu,1,\Delta}(\xi) - U_{\mu,0,\Delta}(\xi)\|$ on (μ, Δ) for ξ in the interval $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$. By setting $\zeta = \Delta\xi$ this reduces to consideration of the differences

$$\begin{aligned} D_1(\mu, \zeta) &= \left| e^{i\zeta} \left(\cos \sqrt{|\mu|^2 + \zeta^2} - \frac{i\zeta \sin \sqrt{|\mu|^2 + \zeta^2}}{\sqrt{|\mu|^2 + \zeta^2}} \right) - \cos |\mu| \right| \\ D_2(\mu, \zeta) &= \left| \frac{\mu \sin \sqrt{|\mu|^2 + \zeta^2}}{\sqrt{|\mu|^2 + \zeta^2}} - \frac{\mu \sin |\mu|}{|\mu|} \right|, \end{aligned} \quad (61)$$

for $\zeta \in [-\pi, \pi]$. As the plots in figures 1 and 2 show, the only way to make both differences small is to take $|\mu|$ small.

In terms of pulse design, this means that Δ should be taken to be small which in

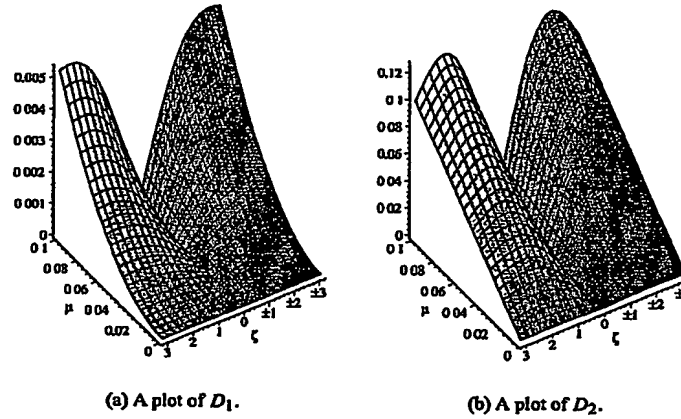


Figure 2: Plots of the error functions D_1 and D_2 for $\mu \in [0, 0.1]$.

turn forces N to be large. A computation with Taylor series shows that

$$\begin{aligned} D_1(\mu, \zeta) &= |\mu|^2 F(\mu, \zeta), \\ D_2(\mu, \zeta) &= |\mu|(|\mu|^2 G(\mu, \zeta) + \zeta^2 H(\zeta^2)), \end{aligned} \quad (62)$$

here F, G, H are entire functions. For a fixed target magnetization r the (μ_j) appearing in (59) satisfy

$$|\mu_j| = O(\Delta) = O(N^{-1}).$$

For ξ restricted to a fixed finite interval, ζ also behaves like $O(N^{-1})$. With this restriction on ξ both error functions behave like $O(N^{-2})$ as N tends to infinity.

We express $U_{1,\Delta}$ as

$$U_{1,\Delta} = U_{\mu_N,0,\Delta}(\text{Id}_2 + E_N)U_{\mu_{N-1},0,\Delta}(\text{Id}_2 + E_{N-1}) \cdots U_{\mu_1,0,\Delta}(\text{Id}_2 + E_1),$$

where

$$\text{Id}_2 + E_j = U_{\mu_j,0,\Delta}^{-1} U_{\mu_j,1,\Delta}.$$

Using this expression and the estimates above one can show that the differences

$$|A_1(\xi) - A_0(\xi)|, \quad |B_1(\xi) - B_0(\xi)|$$

can be made uniformly $O(N^{-1})$ for ξ belonging to a fixed interval, provided N is sufficiently large. Note that if the error in the individual terms were of order N^{-1} then this statement would not be true, as indicated by the well known formula

$$\lim_{N \rightarrow \infty} \left(1 + \frac{1}{N}\right)^N = e.$$

8 ANALYSIS OF THE HARD PULSE APPROXIMATION

25

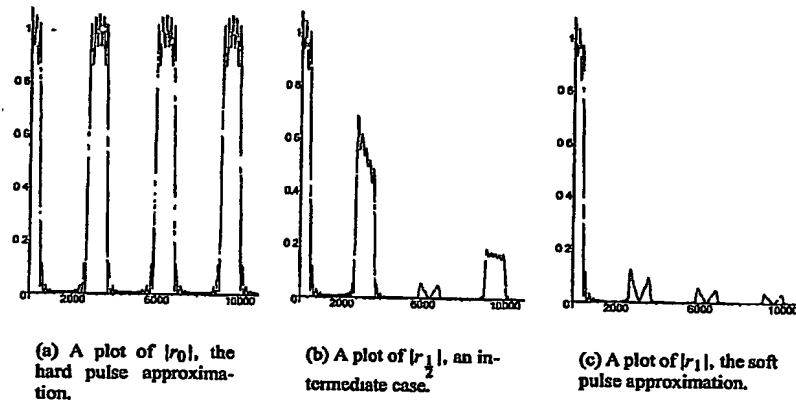


Figure 3: Plots of the absolute magnetization profiles for various approximations to a 90° -sinc pulse showing several fundamental periods.

Thus by taking N large and therefore Δ small the softened SLR pulse should provide a good approximation to the desired magnetization over a fixed interval where A_0 does not vanish. For a fixed N the behavior of $r_1(\xi)$ for large $|\xi|$ is governed by (65). Empirically it is well known that r_1 is quite small outside of $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$. The plots in figure 3 show the magnitude of the magnetization profiles for several approximations to a 90° -sinc-pulse with a bandwidth of 1000Hz. Note that $r_{1/2}$ has considerable support outside the fundamental period $[-500, 500]$.

The answer to the last question is not immediately clear. According to [10], the minimum energy SLR pulse is attained by placing all the roots of P inside the unit circle. This accords well with our analysis, as it is zeros of a in the upper half plane, which corresponds to the exterior of the unit circle, that increase the energy requirements of an RF-envelope. The asymptotic analysis above shows that zeros of A_0 in the upper half plane that are close to the real segment $[-\Delta^{-1}\pi, \Delta^{-1}\pi]$ are likely to produce zeros of A_1 in the upper half plane which thereby increase the energy requirement of the softened pulse. The behavior of the zeros of A_ϵ as ϵ goes from 0 to 1 seems an interesting question for further investigation.

By way of contrast, using the IST method one is in direct control of the zeros of a in the upper half plane as well as the phase of the magnetization profile. We close this section by stating an estimate which shows that replacing the smooth RF-envelope found using the IST by a soft pulse approximation g_{s1} does not result in substantial errors in the magnetization profile.

Theorem 8. Suppose that h_1 and h_2 are integrable potentials for the ZS-system, and

that $F_i^\pm(\xi)$, $i = 1, 2$ solves the ZS-system with potential h_i , $i = 1, 2$. Let

$$\begin{aligned} H^-(t) &= \int_{-\infty}^t |h_1(s) - h_2(s)| ds, & Q^-(t) &= \int_{-\infty}^t |h_1(s)| ds, \\ H^+(t) &= \int_t^{\infty} |h_1(s) - h_2(s)| ds, & Q^+(t) &= \int_t^{\infty} |h_1(s)| ds. \end{aligned} \quad (63)$$

If

$$\lim_{t \rightarrow \pm\infty} \|F_1^\pm(\xi; t) - F_2^\pm(\xi; t)\| = 0.$$

then, for $\xi \in \mathbb{R}$ the differences satisfy the estimates

$$\|F_1^\pm(\xi; t) - F_2^\pm(\xi; t)\| \leq 2I_0(2\|h_2\|_{L^1})H^\pm(t)\exp(Q^\pm(t)). \quad (64)$$

The proof is in Appendix B.

In light of these estimates it is clear that by choosing a soft approximation q_{s1} to the smooth pulse q_s so that $\|q_s - q_{s1}\|_{L^1}$ is small, the reflection coefficient r_{s1} can be made uniformly close to r_s over any desired interval where a_s does not vanish. Once again this amounts to using a small value of Δ in the definition of q_{s1} .

9 Appendices

9.1 Appendix A: Solving the Marchenko equation II

We now explain the modifications to the algorithm presented in section 5 which are required if the intermediate potentials $\{q_j\}$ have non-trivial bound states. Suppose that φ is a function with two continuous derivatives and $-\infty < \alpha < \beta < \infty$. In Appendix B we establish the following asymptotic formulae and estimates for the scattering data (a_c, b_c) for the ZS-system with potential $\chi_{[\alpha, \beta]}\varphi$:

$$\begin{aligned} a_c(\xi) &= 1 + \frac{\|\varphi\|_{L^2[\alpha, \beta]}^2}{2i\xi} + O\left(\frac{1}{(1+|\xi|)^2}\right), \\ e^{2i\xi\beta}b_c(\xi) &= e^{2i\xi\beta}\hat{T}_{[\alpha, \beta; \varphi^*(\alpha), \varphi^*(\beta)]}(2\xi) + O\left(\frac{1}{(1+|\xi|)^2}\right). \end{aligned} \quad (65)$$

Here $T_{[\mu, \nu; A, B]}$ is the piecewise, linear function

$$T_{[\mu, \nu; A, B]}(t) = \chi_{[\mu, \nu]}(t) \frac{B(t - \mu) + A(\nu - t)}{\nu - \mu}. \quad (66)$$

These formulae hold uniformly in the upper half plane with the implied constants in the O -terms bounded in terms of the $\mathcal{E}^2([\alpha, \beta])$ -norm of φ . The formula for a_c shows that the zeros of this function, in the upper half plane are constrained to lie in a fixed ball

Since the potentials $\{q_j\}$ have bounded support and q is assumed to have at least two derivatives the estimates in (65) apply for each $j \geq 1$. The scattering coefficients $\{(a_j, b_j)\}$ are entire functions. The asymptotic formula for a_j implies that it has at most finitely zeros in the upper half plane. Denote these zeros by $\{\xi_i^{(j)} : i = 1, \dots, N_j\}$. At step j we have determined

$$\tilde{q}_j = \chi_{(-\infty, t_-^{(j-1)})} q + \chi_{[t_+^{(j-1)}, \infty)} q.$$

Equation (39) implies that

$$\int_{-\infty}^{\infty} |q(t)|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log(1 + |r(\xi)|^2) d\xi.$$

It is not difficult to show that (39) can be extended to potentials of the form $\chi_{[\alpha, \beta]} q$, with q twice differentiable, so that:

$$\int_{-\infty}^{\infty} |q_j(t)|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log(1 + |r_j(\xi)|^2) d\xi + 4 \sum_{i=1}^{N_j} \text{Im } \xi_i^{(j)}.$$

Using these formulae gives

$$4 \sum_{i=1}^{N_j} \text{Im } \xi_i^{(j)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left(\frac{1 + |r(\xi)|^2}{1 + |r_j(\xi)|^2} \right) d\xi - \int_{-\infty}^{\infty} |\tilde{q}_j(t)|^2 dt. \quad (70)$$

This relation gives an upper bound on the imaginary part of any root of a_j :

$$\begin{aligned} 4 \text{Im}(\xi_i^{(j)}) &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left(\frac{1 + |r(\xi)|^2}{1 + |r_j(\xi)|^2} \right) d\xi - \int_{-\infty}^{\infty} |\tilde{q}_j(t)|^2 dt \\ &\leq \int_{-\infty}^{\infty} |q(t)|^2 dt. \end{aligned} \quad (71)$$

We fix η to be a non-negative number, larger than $\max_{i,j} \{\text{Im } \xi_i^{(j)}\}$ or zero if this maximum is zero.

In light of the upper bound (71) on the imaginary parts of the zeros of the functions $\{a_j\}$, we can use the estimates in (65) to further conclude that the integrands $\{f_j : j \geq 1\}$ are uniformly bounded in $[2t_-^{(j-1)}, 2t_+^{(j-1)}]$. Indeed each is the sum of a piecewise linear function of the type defined in (66) and a continuous function with support in $(-\infty, 2t_+^{(j-1)}]$ which vanishes at $-\infty$. In light of Theorem 5 this establishes the claim, made in section 5, that there is a $\Delta > 0$ so that

$$|t_{\pm}^{(j)} - t_{\pm}^{(j-1)}| > \Delta.$$

These observations show that the algorithm in section 5 applies to the case where the intermediate potentials have bound states so long as we use (68) ((69)) to compute the $\{f_j\}$ ($\{g_j\}$). This formula requires the analytic continuation of the scattering data into the upper half plane. Since we know, *a priori* that a_j and b_j have such extensions, the estimates in (65) apply to show that the Cauchy integral formula gives:

$$\begin{aligned} a_j(\xi + i\eta) &= 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(a_j(\zeta) - 1)d\zeta}{\zeta - (\xi + i\eta)}, \\ b_j(\xi + i\eta) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b_j(\zeta)d\zeta}{\zeta - (\xi + i\eta)}. \end{aligned} \quad (72)$$

To obtain an algorithm that succeeds in solving the Marchenko equations in all cases we need only modify Step 1 in the earlier algorithm:

STEP 1': Use (28) to determine if the reflection coefficient for q_j has poles in the upper half plane. If so extend (a_j, b_j) to the line $\text{Im } \xi = \eta_0$ and use (68) ((69)) to determine the kernel function(s) f_j (g_j) supported in the ray $(-\infty, 2t_+^{(j-1)})$ ($[2t_-^{(j-1)}, \infty)$).

We close this appendix with some remarks on the implementation of this algorithm.

1. It is well known that it is not necessary to solve the 2×2 -matrix system (37) to find \mathcal{H}_{2j} . By substitution this can be reduced to solving a scalar integral equation, see [1].
2. Given that the initial data r (or m^∞) is assumed to be smooth, with bounded support there is no problem, in principle, in determining $q_\pm^{(0)}$. The scattering data for these potentials can be determined by numerically solving the ZS-system with the given potentials. The method used in section 8 gives alternate algebraic approach to the approximate determination of the scattering coefficients.
3. In the subsequent steps of the algorithm we are working with potentials with bounded support but jump discontinuities. The difficulties that this might engender, like the Gibbs phenomenon, can be avoided by using the asymptotic expansion (67) in (68). That is, we set

$$\check{r}_j(\xi) = r_j(\xi) - \hat{T}_{[t_-^{(j-1)}, t_+^{(j-1)}; q_+^{(j-1)}, q_-^{(j-1)}]}(2\xi).$$

The contribution of $\hat{T}_{[t_-^{(j-1)}, t_+^{(j-1)}; q_+^{(j-1)}, q_-^{(j-1)}]}$ to f_j can be exactly determined. It is a function with support in $[2t_-^{(j-1)}, 2t_+^{(j-1)}]$. The function \check{r}_j is $O(|\xi|^{-2})$ as $|\xi|$ goes to infinity. This estimate holds uniformly in strips with bounded imaginary part in the upper half space. Thus in the computation of the approximate Fourier inverses

$$\frac{1}{2\pi} \int_{-R}^R r_j(\xi + i\eta_0) e^{i(\xi + i\eta_0)t} d\xi$$

we can avoid bothersome Gibbs-type artifacts. If q is smooth then there is a complete asymptotic expansion for each r_j which only depends on the (in principle known) values of the derivatives $\{q^{(k)}(t_{\pm}^{(j-1)})\}$. Using further terms in this expansion can produce even more rapidly convergent correction terms. This should be done with care as it requires the computation of derivatives of q which could in turn lead to numerical instability.

4. After Step 0 we are looking for spatially limited potentials. The input data for Step 1 is essentially in the frequency domain. Since we know $[t_-^{(0)}, t_+^{(0)}]$ we can choose the sample spacing in the frequency domain so as to avoid aliasing errors in the construction of these potentials.
5. The Cauchy integral in (72) is a shift invariant filter with transfer function $2\pi i \chi_{(-\infty, 0]}(t) e^{t\eta}$. As such it is easily implemented in practice.
6. After a potential q with reflection coefficient r and no bound states is found, it is a relatively simple matter to add in bound states. An algorithm to do this is described in section 6 of [3]. One possible usage for bound states is to reduce the effective support of the potential. To that end one might choose points $\{\xi_1, \dots, \xi_N\}$ in the upper half plane and norming constants $\{C_j\}$ so that sum

$$i \sum_{j=1}^N C_j e^{i\xi_j t}$$

approximates

$$\tilde{r}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(\xi) e^{i\xi t} d\xi$$

over some ray $[T, \infty)$. With these choices the effective support of f defined in (36) can be made smaller than that of \tilde{r} and thereby a potential with smaller (effective) support, producing the same magnetization profile can be found. This is of course at the expense of

$$4 \sum_{j=1}^N \text{Im } \xi_j$$

in additional energy. In section 6 we saw that the effective support of q determines the rephasing which is needed to obtain the specified magnetization profile.

7. While the bound states do not affect the observable magnetization they do affect the spins themselves. Bound states might prove useful in designing pulse sequences intended to manipulate higher order quantum coherences.
8. The computation in (28) has an integer valued result. Using the asymptotic

formula (65) it is not difficult to give an estimate for R so that the finite integral

$$\int_{-R}^R \frac{a_j'(\xi) d\xi}{a_j(\xi) \cdot (2\pi i)}$$

suffices to determine N_j .

9.2 Appendix B: Mathematical proofs

The Paley-Wiener Theorems: The "Paley-Wiener" Theorem is used several times in this paper. This is really a collection of results relating the support properties of a function defined on \mathbb{R} to the analytic continuation of its Fourier transform. A complete discussion can be found in [17]. Two results adequate for our applications are:

Theorem 9 (Paley-Wiener Theorem I). *A function f in $L^2(\mathbb{R})$ has bounded support in the interval $[-L, L]$ if and only if its Fourier transform has an analytic extension to the entire complex plane satisfying:*

$$|\hat{f}(\xi + i\eta)| \leq Ce^{L|\eta|}, \quad (73)$$

for a constant C .

and

Theorem 10 (Paley-Wiener Theorem II). *A function f in $L^2(\mathbb{R})$ has support in the ray $(-\infty, 0]$ if and only if its Fourier transform has an analytic extension to the upper half plane satisfying*

$$\begin{aligned} \int_{-\infty}^{\infty} |\hat{f}(\xi + i\eta)|^2 d\xi &\leq C \text{ and} \\ \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} |\hat{f}(\xi + i\eta) - \hat{f}(\xi)|^2 d\xi &= 0. \end{aligned} \quad (74)$$

The proof of Theorem 5: Assuming that $I_f(2t) < 1$ We need to show that the Marchenko equation,

$$\mathcal{K}_t(s) - \int_t^\infty \mathcal{K}_t(s+x) \mathcal{F}(t+x) dx = -\mathcal{F}(s+t), \quad (75)$$

can be solved by the iteration defined in (44) and the solution satisfies the estimate in (45). The proof is a simple induction argument. Clearly $\|\mathcal{K}_t^0(s)\| \leq M_f(t+s)$. Assume that

$$\|\mathcal{K}_t^j(s)\| \leq M_f(t+s) [1 + I_f(2t) + \dots + I_f^j(2t)]. \quad (76)$$

It follows from this assumption that

$$\|\mathcal{H}_t^{j+1}(s)\| \leq M_f(t+s) + \left[1 + I_f(2t) + \dots + I_f^j(2t)\right] \times \int_t^\infty M_f(t+x) \|\mathcal{F}(t+x)\| dx.$$

The induction hypothesis (76), with j replaced by $j+1$ follows easily from this estimate. Using a similar argument we can show that, for $j \geq 1$

$$\|\mathcal{H}_t^j(s) - \mathcal{H}_t^{j-1}(s)\| \leq M_f(t+s) I_f(2t)^j \quad (77)$$

If $I_f(2t) < 1$ then this estimate shows that $\{\mathcal{H}_t^j\}$ is a uniformly convergent sequence for $s \in [t, \infty)$. The limit, \mathcal{H}_t^∞ is evidently a solution to (75). We can also pass to the limit in (76) to conclude that \mathcal{H}_t^∞ satisfies (45).

The proof of (65): We use the integral equations (16) defining the components of ψ_{1-} . Let

$$f_1(\xi; t) = \psi_{1-}(\xi; t) e^{\xi t}.$$

It follows from (15), (16) and (19) that

$$\begin{aligned} a_c(\xi) &= 1 + \int_\alpha^\beta M_1(\xi; \infty, s) f_1(\xi; s) ds, \\ b_c(\xi) &= - \int_\alpha^\beta e^{-2i\xi s} \varphi^*(s) f_1(\xi; s) ds. \end{aligned} \quad (78)$$

In light of the simple form assumed by the potential we see, integrating by parts, that

$$\begin{aligned} M_1(\xi; \infty, s) &= -\varphi^*(s) \chi_{[\alpha, \beta]}(s) \int_s^\beta e^{2i\xi(x-s)} \varphi(x) dx \\ &= -\varphi^*(s) \chi_{[\alpha, \beta]}(s) \left[\frac{e^{2i\xi(\beta-s)} \varphi(\beta) - \varphi(s)}{2i\xi} - \int_s^\beta \frac{e^{2i\xi(x-s)} \varphi'(x)}{2i\xi} dx \right] \end{aligned} \quad (79)$$

Since $(x-s) \geq 0$, in the integral, this expression remains uniformly bounded for ξ in the upper half plane and implies that

$$M_1(\xi; \infty, s) = -\varphi^*(s) \chi_{[\alpha, \beta]}(s) \left[\frac{e^{2i\xi(\beta-s)} \varphi(\beta) - \varphi(s)}{2i\xi} + O\left(\frac{1}{1+|\xi|^2}\right) \right]. \quad (80)$$

This formula holds for ξ with non-negative real part with the implied constant in the O -term only depending on the $\mathcal{C}^2[\alpha, \beta]$ -norm of φ .

9 APPENDICES

33

Using this formula for $M_1(\xi; \infty, s)$ in (78) gives

$$\begin{aligned} a_c(\xi) &= 1 + \frac{\|\varphi X_{(\alpha, \beta)}\|_{L^2}^2}{2i\xi} + O\left(\frac{1}{1+|\xi|^2}\right) \\ b_c(\xi) &= -\int_{\alpha}^{\beta} e^{-2i\xi s} \varphi^*(s) ds + O\left(\frac{1}{1+|\xi|^2}\right). \end{aligned} \quad (81)$$

This completes the analysis of a_c ; for b_c we integrate by parts again to get:

$$b_c(\xi) = \frac{e^{-2i\xi\beta} \varphi^*(\beta) - e^{-2i\xi\alpha} \varphi^*(\alpha)}{2i\xi} + O\left(\frac{1}{1+|\xi|^2}\right). \quad (82)$$

The proof is completed by observing that

$$\hat{T}_{[\mu, \nu; A, B]} = -\frac{B e^{-i\nu\xi} - A e^{-i\mu\xi}}{i\xi} + \frac{B - A}{\beta - \alpha} \frac{e^{-i\nu\xi} - e^{-i\mu\xi}}{\xi^2}.$$

If φ has $k+1$ derivatives then we can integrate by parts in (79) k -times to get an asymptotic expansion for M_1 with an error term of order $O(|\xi|^{-(k+1)})$. This can in turn be used in (16) and (78) to obtain higher order asymptotic expansions for a_c and b_c . For smooth potentials, vanishing at infinity, expansions of this type are well known in the inverse scattering literature.

The proof of Theorem 8: The proof of this result is a small modification of the proof of the estimate (18) given in [1]. Let σ denote the 2×2 -matrix

$$\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and let

$$D^{\pm}(\xi; t) = e^{-it\xi\sigma} (F_1^{\pm} - F_2^{\pm}).$$

For real ξ and t the matrices $e^{\pm it\xi\sigma}$ are unitary and therefore

$$\|F_1^{\pm} - F_2^{\pm}\| = \|D^{\pm}\|.$$

The vector valued functions D^{\pm} satisfy the equations

$$\begin{aligned} \partial_t D^{\pm} &= e^{-it\xi\sigma} \begin{bmatrix} 0 & h_1 \\ -h_1^* & 0 \end{bmatrix} e^{it\xi\sigma} D^{\pm} + e^{-it\xi\sigma} \begin{bmatrix} 0 & (h_1 - h_2) \\ -(h_1^* - h_2^*) & 0 \end{bmatrix} F_2^{\pm}. \\ \lim_{t \rightarrow \pm\infty} D^{\pm}(\xi; t) &= 0. \end{aligned} \quad (83)$$

We now restrict our attention to the $-$ case, the $+$ case is handled by an essentially identical argument.

The differential equation in (83) is equivalent to the integral equation

$$\begin{aligned} D^-(\xi; t) = & \int_{-\infty}^t e^{-is\xi\sigma} \begin{bmatrix} 0 & h_1(s) \\ -h_1^*(s) & 0 \end{bmatrix} e^{is\xi\sigma} D^-(\xi; s) ds + \\ & \int_{-\infty}^t e^{-is\xi\sigma} \begin{bmatrix} 0 & (h_1 - h_2)(s) \\ -(h_1^* - h_2^*)(s) & 0 \end{bmatrix} F_2^-(s) ds. \end{aligned} \quad (84)$$

This is a Volterra equation which can be solved by the following iteration:

$$\begin{aligned} D_0^-(\xi; t) &= \int_{-\infty}^t e^{-is\xi\sigma} \begin{bmatrix} 0 & (h_1 - h_2)(s) \\ -(h_1^* - h_2^*)(s) & 0 \end{bmatrix} F_2^-(s) ds \\ D_j^-(\xi; t) &= D_0^-(\xi; t) + \int_{-\infty}^t e^{-is\xi\sigma} \begin{bmatrix} 0 & h_1(s) \\ -h_1^*(s) & 0 \end{bmatrix} e^{is\xi\sigma} D_{j-1}^-(\xi; s) ds, \end{aligned} \quad (85)$$

for $j \geq 1$.

First observe that the operator norm defined by the Euclidean norm satisfies

$$\left\| \begin{bmatrix} 0 & \alpha \\ -\alpha^* & 0 \end{bmatrix} \right\| = |\alpha|$$

and therefore (18) implies that

$$\|D_0^-(\xi; t)\| \leq 2I_0(\|h_2\|_{L^1})H^-(t);$$

we make the inductive assumption that

$$\|D_j^-(\xi; t)\| \leq 2I_0(\|h_2\|_{L^1})H^-(t) \left[\sum_{k=0}^j \frac{(Q^-(t))^k}{k!} \right]. \quad (86)$$

From these assumptions and (85) it follows that

$$\|D_{j+1}^-(\xi; t)\| \leq 2I_0(\|h_2\|_{L^1}) \left(H^-(t) + \int_{-\infty}^t |h_1(s)| H^-(s) \left[\sum_{k=0}^j \frac{(Q^-(s))^k}{k!} \right] ds \right). \quad (87)$$

The definition of Q^- implies that

$$|h_1(s)| = \partial_s Q^-(s)$$

and therefore (87) implies that

$$\begin{aligned} \|D_{j+1}^-(\xi; t)\| &\leq 2I_0(\|h_2\|_{L^1})H^-(t) \left[1 + \int_{-\infty}^t \partial_s \left(\sum_{k=0}^j \frac{(Q^-(s))^{k+1}}{(k+1)!} \right) ds \right] \\ &= 2I_0(\|h_2\|_{L^1})H^-(t) \left(\sum_{k=0}^{j+1} \frac{(Q^-(t))^k}{k!} \right) \end{aligned} \quad (88)$$

REFERENCES

35

thus completing the proof of the induction step. As

$$\lim_{j \rightarrow \infty} D_j^- = D^-,$$

letting j tend to infinity completes the proof of the theorem.

References

- [1] M. ABLOWITZ, D. KAUP, A. NEWELL, AND H. SEGUR, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Applied Math., 53 (1974), pp. 249-315.
- [2] R. BEALS AND R. COIFMAN, *Scattering and inverse scattering for first order systems*, CPAM, 37 (1984), pp. 39-90.
- [3] ———, *Scattering and inverse scattering for first order systems: II*, Inverse Problems, 3 (1987), pp. 577-593.
- [4] R. BEALS, P. DEIFT, AND C. TOMEI, *Direct and inverse scattering on the line*, American Mathematical Society, Providence, 1988.
- [5] J. CARLSON, *Exact solutions for selective-excitation pulses*, Jour. of Mag. Res., 94 (1991), pp. 376-386.
- [6] ———, *Exact solutions for selective-excitation pulses. II. Excitation pulses with phase control*, Jour. of Mag. Res., 97 (1992), pp. 65-78.
- [7] L. FADDEEV AND L. TAKHTAJAN, *Hamiltonian Methods in the Theory of Solitons*, Springer Verlag, Berlin, Heidelberg, New York, 1987.
- [8] F. GRÜNBAUM AND A. HASENFELD, *An exploration of the invertibility of the Bloch transform*, Inverse Problems, 2 (1986), pp. 75-81.
- [9] H. E. MOSES AND R. T. PROSSER, *Eigenvalues and eigenfunctions associated with the Gel'fand-Levitan equation*, J. Math. Phys., 25 (1984), pp. 108-112.
- [10] J. PAULY, P. L. ROUX, D. NISHIMURA, AND A. MACOVSKI, *Parameter relations for the Shinnar-Le Roux selective excitation pulse design algorithm*, IEEE Trans. on Med. Imaging, 10 (1991), pp. 53-65.
- [11] D. E. ROURKE AND P. G. MORRIS, *The inverse scattering transform and its use in the exact inversion of the Bloch equation for noninteracting spins*, Jour. of Mag. Res., 99 (1992), pp. 118-138.
- [12] P. L. ROUX, *Exact synthesis of radio frequency waveforms*, in Proceeding of 7th annual meeting of the SMRM, 1988, p. 1049.
- [13] W. RUDIN, *Real and Complex Analysis*, McGraw Hill, New York, 1966.
- [14] M. SHINNAR, L. BOLINGER, AND J. LEIGH, *The synthesis of soft pulses with a specified frequency response*, Mag. Res. in Med., 12 (1989), pp. 88-92.
- [15] M. SHINNAR, S. ELEFF, H. SUBRAMANIAN, AND J. LEIGH, *The synthesis of pulse sequences yielding arbitrary magnetization vectors*, Mag. Res. in Med., 12 (1989), pp. 74-88.
- [16] M. SHINNAR AND J. LEIGH, *The application of spinors to pulse synthesis and analysis*, Mag. Res. in Med., 12 (1989), pp. 93-98.
- [17] E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Press, Princeton, NJ, 1971.

REFERENCES

36

- [18] H. WIDOM, *Lectures on Integral Equations*, Van Nostrand; Reinhold Co., New York-Toronto, Ont.-London, 1969.
- [19] V. ZAKHAROV AND L. FADDEEV, *Korteweg-de Vries equation, a completely integrable Hamiltonian system*, Funk. Anal. Prilöz., 5 (1971), pp. 18-27.
- [20] V. ZAKHAROV AND S. MANAKOV, *On the complete integrability of the non-linear Schrödinger equation*, Teor. Mat. Fiz., 19 (1974), pp. 332-343.

I CLAIM:

1. A method of synthesizing minimum energy pulses for a given magnetization profile in a magnetic resonance imaging system, comprising the steps of:
 determining minimum energy for an RF envelope producing the given magnetization profile;
 using an inverse scattering transform to determine a minimum energy smooth pulse RF envelope that approximates the energy of said minimum energy RF envelope; and
 generating pulses with energy that approximates said minimum energy smooth pulse RF envelope.
2. A method as in claim 1, comprising the step of determining auxiliary parameters including bound states and norming constants for reduced scattering data of a potential for the minimum energy used in said inverse scattering transform.
3. A method as in claim 1, wherein the energy required for the minimum energy smooth pulse RF envelope is independent of the phase of transverse magnetization.
4. A method as in claim 1, comprising the additional step of increasing the energy of the smooth pulse RF envelope in order to reduce rephrasing time.
5. A method as in claim 2, wherein the step of determining the minimum energy smooth pulse RF envelope comprises the step of solving left and right Marchenko equations by iteration on maximal intervals to determine the reduced scattering data and iteratively removing parts of the reduced scattering data produced by already known parts of the potential.
6. A method as in claim 1, comprising the step of replacing the smooth pulse RF envelope with a soft pulse approximation to said smooth pulse RF envelope.
7. A method of generating a desired frequency dependent excitation in a magnetic resonance imaging system using minimum energy selective RF pulses for a given magnetization profile corresponding to said desired frequency dependent

excitation, comprising the steps of:

determining minimum energy for an RF envelope producing the given magnetization profile;

using an inverse scattering transform to determine a minimum energy smooth pulse RF envelope that approximates the energy of said minimum energy RF envelope;

generating pulses with energy that approximates said minimum energy smooth pulse RF envelope; and.

applying the pulses to said magnetic resonance imaging system to obtain a desired selective excitation.

**This Page is Inserted by IFW Indexing and Scanning
Operations and is not part of the Official Record**

BEST AVAILABLE IMAGES

Defective images within this document are accurate representations of the original documents submitted by the applicant.

Defects in the images include but are not limited to the items checked:

- ☐ BLACK BORDERS
- ☐ IMAGE CUT OFF AT TOP, BOTTOM OR SIDES
- ☐ FADED TEXT OR DRAWING
- ☐ BLURRED OR ILLEGIBLE TEXT OR DRAWING
- ☐ SKEWED/SLANTED IMAGES
- ☐ COLOR OR BLACK AND WHITE PHOTOGRAPHS
- ☐ GRAY SCALE DOCUMENTS
- ☒ LINES OR MARKS ON ORIGINAL DOCUMENT
- ☒ REFERENCE(S) OR EXHIBIT(S) SUBMITTED ARE POOR QUALITY
- ☐ OTHER: _____

IMAGES ARE BEST AVAILABLE COPY.

As rescanning these documents will not correct the image problems checked, please do not report these problems to the IFW Image Problem Mailbox.